

ON THE GROWTH PROPERTIES OF A FUNCTION OF TWO COMPLEX VARIABLES GIVEN BY ITS POWER SERIES EXPANSION

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1. Introduction. One of the most fundamental formulas in the theory of functions of one complex variable is the Cauchy integral formula. It is of particular value in the Weierstrass-Hadamard approach, i.e., in obtaining properties of a function from the coefficients of its power series expansion. A similar formula cannot be obtained for functions of two complex variables for an arbitrary four-dimensional domain, as is obtained, for instance, for the bicylinder, where the integration is taken over a two-dimensional surface on the boundary. Bergman⁽¹⁾ has shown, however, that for certain domains far more general than those previously considered, i.e., domains bounded by a finite number of analytic hypersurfaces, an analogous formula does exist, the double integral being taken essentially over the two-dimensional surface common to two or more of the analytic bounding hypersurfaces⁽²⁾.

In this paper we shall obtain growth properties in terms of the coefficients of the power series expansion of a function $f(z_1, z_2)$ of two complex variables analytic in special domains of the type mentioned above; first, with the aid of Bergman's integral formula, along the two-dimensional surfaces common to the bounding hypersurfaces, and then, along a class of two-dimensional surfaces lying in only one of the bounding hypersurfaces and having a line of contact with another bounding hypersurface. We also obtain a mapping theorem which determines from the coefficients a convex region in the f_1f_2 -plane, $f(z_1, z_2) = f_1 + if_2$, which must be contained in the smallest convex region of the mapping on the f_1f_2 -plane of the surfaces considered.

2. Properties of f associated with $G^2(r)$. Let us consider a finite four-dimensional domain $\mathfrak{M}^4$ which is bounded by the hypersurfaces

$$(2.1) \quad \begin{aligned} s_1^3(r) &\equiv E[z_2 = re^{i\lambda_1}, 0 \leq \lambda_1 \leq 2\pi], \\ s_2^3(r) &\equiv E[z_1 = re^{i\lambda_2} + p(\lambda_2)z_2 \equiv h(\lambda_2, z_2), 0 \leq \lambda_2 \leq 2\pi], \end{aligned}$$

and which depends on a positive parameter r ; $p(\lambda_2)$ is assumed merely to have

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(¹) Bergman [2, 3]. See the bibliography at the end of this paper.

(²) Bergman calls such surfaces "distinguished boundary surfaces."

a first derivative. Let $G^2(r)$ be the two-dimensional surface on the boundary of $\mathfrak{M}^4$ which is the common part of the bounding hypersurfaces, i.e.,

$$(2.2) \quad G^2(r) \equiv s_1^3 \cdot s_2^3.$$

THEOREM I. *Given a function $f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$ regular in the domain $\overline{\mathfrak{M}^4}(r)$; if $M(r)$ is the maximum-modulus of $f(z_1, z_2)$ on $G^2(r)$, then*

$$(2.3) \quad M(r) \geq \max_{m,n} \frac{r^{m+n} |a_{mn}|}{G(m, n; p) B(p)},$$

where m and n range over all non-negative integral values, $B(p)$ is a constant depending upon p , and $G(m, n; p)$ is a function of m, n , and p , given by

$$(2.4) \quad \begin{aligned} & 1 + \int_0^{n+1} \left(1 + x \frac{1 + \log m}{m} \right)^m \max |p(\lambda_2)|^x dx - \frac{m}{\log p} - \frac{m}{1 + \log m}, \\ & \text{when } \max |p| < 1, m \geq 1, \\ & 1 + \int_0^{n+1} \left(1 + x \frac{1 + \log m}{m} \right)^m \max |p(\lambda_2)|^x dx, \\ & \text{when } \max |p| \geq 1, m \geq 1, \\ & \frac{1 - \max |p(\lambda_2)|^{n+1}}{1 - \max |p(\lambda_2)|}, \quad \text{when } m = 0 \text{ for all } p. \end{aligned}$$

Proof of Theorem I. Keeping z_2 constant, say equal to t_2 , we obtain for a particular value of z_1 , say t_1 ,

$$(2.5) \quad f(t_1, t_2) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f[h(\lambda_2, t_2), t_2] [ire^{i\lambda_2} + p'(\lambda_2)t_2] d\lambda_2}{[(re^{i\lambda_2} + p(\lambda_2)t_2) - t_1]}.$$

Since the numerator of the integrand is an analytic function of t_2 , we again apply the Cauchy integral formula and obtain

$$(2.6) \quad \begin{aligned} f(t_1, t_2) = & \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f[h(\lambda_2, re^{i\lambda_1}), re^{i\lambda_1}]}{[(re^{i\lambda_2} + p(\lambda_2)t_2) - t_1][re^{i\lambda_1} - t_2]} \\ & \cdot [ire^{i\lambda_2} + p're^{i\lambda_1}] ire^{i\lambda_2} d\lambda_1 d\lambda_2. \end{aligned}$$

For the m th derivative of $f(t_1, t_2)$ with respect to t_1 , we obtain

$$(2.7) \quad \frac{\partial^m f(t_1, t_2)}{\partial t_1^m} = \frac{m!}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f[h(\lambda_2, re^{i\lambda_1}), re^{i\lambda_1}]}{[(re^{i\lambda_2} + p(\lambda_2)t_2) - t_1]^{m+1} [re^{i\lambda_1} - t_2]} \cdot [ire^{i\lambda_2} + p'(\lambda_2)re^{i\lambda_1}] ire^{i\lambda_2} d\lambda_1 d\lambda_2.$$

Let

$$H_1 \equiv (re^{i\lambda_2} + p(\lambda_2)t_2) - t_1, \quad H_2 \equiv re^{i\lambda_1} - t_2.$$

For the n th derivative of $1/H_1^{m+1}H_2$ with respect to t_2 , we obtain by Leibnitz' rule

$$(2.8) \quad \left[1 + \sum_{\nu=1}^n \frac{(m+\nu)!}{m!\nu!} \left(\frac{H_2}{H_1} p(\lambda_2) \right)^\nu \right] \frac{n!}{H^{m+1}H_2^{n+1}}.$$

Hence we obtain for $\partial^{m+n}f(t_1, t_2)/\partial t_1^m \partial t_2^n$ the expression

$$(2.9) \quad \frac{m!n!}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f[h, \lambda_1] [ire^{i\lambda_2} + p'(\lambda_2)re^{i\lambda_1}] ire^{i\lambda_1}}{[(re^{i\lambda_2} + p(\lambda_2)t_2) - t_1]^{m+1} [re^{i\lambda_1} - t_2]^{n+1}} \\ \cdot \left[1 + \sum_{\nu=1}^n \frac{(m+\nu)!}{m!\nu!} \left(\frac{H_2}{H_1} p \right)^\nu \right] d\lambda_1 d\lambda_2.$$

Now

$$a_{mn} = \frac{\partial^{m+n}f(0, 0)}{m!n!\partial t_1^m \partial t_2^n}.$$

Hence

$$(2.10) \quad a_{mn} = \frac{1}{(2\pi i)^2} \int_0^{2\pi} \int_0^{2\pi} \frac{f[h, \lambda_1] [ire^{i\lambda_2} + p're^{i\lambda_1}] ire^{i\lambda_1}}{r^{m+n+2} \exp \{ i(m\lambda_2 + n\lambda_1 + \lambda_2 + \lambda_1) \}} \\ \cdot \left[1 + \sum_{\nu=1}^n \frac{(m+\nu)!}{m!\nu!} (e^{i(\lambda_1 - \lambda_2)} p)^\nu \right] d\lambda_1 d\lambda_2.$$

Taking the absolute value of a_{mn} we get

$$(2.11) \quad |a_{mn}| \leq \frac{1}{4\pi^2} \frac{M(r) \max_{0 \leq \lambda_2 \leq 2\pi} [1 + |p'(\lambda_2)|]}{r^{m+n}} \\ \cdot \left[1 + \sum_{\nu=1}^n \frac{(m+\nu)!}{m!\nu!} (\max |p|^\nu) \right] 4\pi^2.$$

Now for $m \geq 1$, it can be shown that

$$(2.12) \quad 1 + \sum_{\nu=1}^n \left[\frac{(m+\nu)!}{m!\nu!} \max |p(\lambda_2)|^\nu \right] \\ \leq 1 + \sum_{\nu=1}^n \left(1 + \nu \frac{1 + \log m}{m} \right)^m |p|^\nu.$$

When $|p| < 1$ we have

$$(2.13) \quad 1 + \sum_{\nu=1}^n \frac{(m+\nu)!}{m!\nu!} \max |p(\lambda_2)|^\nu \\ \leq 1 + \int_1^n \left[1 + x \frac{1 + \log m}{m} \right]^m \max |p|^x dx - \frac{m}{\log p} - \frac{m}{1 + \log m},$$

and when $|p| \geq 1$,

$$(2.14) \quad 1 + \sum_{\nu=1}^n \frac{(m+\nu)!}{m!\nu!} \max |p|^\nu \leq \int_0^{n+1} \left[1 + x \frac{1 + \log m}{m} \right]^m \max |p|^x dx.$$

When $m=0$, $(1 - \max |p|^{n+1}) / (1 - \max |p|)$ is the exact value of the left-hand side of (2.14) for all p .

Therefore for all differentiable $p(\lambda_2)$ and non-negative integral values of m and n we have

$$(2.15) \quad |a_{mn}| \leq \frac{M(r)B(p)}{r^{m+n}} G(m, n; p),$$

where $B(p) = \max_{0 \leq \lambda_2 \leq 2\pi} (1 + |p'|)$, or

$$(2.16) \quad M(r) \geq \frac{r^{m+n} |a_{mn}|}{B(p)G(m, n; p)}.$$

To find those values of m and n , say $\mu(r)$ and $\nu(r)$, for which the right-hand expression in (2.16) is maximum for a given r , we take the logarithm of the expression, letting $-\log |a_{mn}| = g_{mn}$ and employ a generalized Newton polygon method. Then

$$(2.17) \quad \begin{aligned} g_{mn} - (m+n) \log r + \log B + \log G(m, n) \\ \geq g_{\mu\nu} - (\mu + \nu) \log r + \log B + \log G(\mu, \nu) = C. \end{aligned}$$

We choose m , n , and g_{mn} as the x -, y -, and z -axes, respectively, and plot the points (m, n, g_{mn}) . Then the m and n of the first point which lies in the surface $z = x \log r + y \log r - \log G(x, y) - \log B + k$ as this surface is translated along the z -axis from $-\infty$ by varying k , i.e., until $k = C$, are the μ and ν which give the right-hand side of (2.16) a maximum. If there is more than one point lying on the surface, the one with the smaller m is chosen; if the m 's are the same, the one with the smaller n is chosen. μ and ν are obviously functions of r .

We then have

$$(2.18) \quad M(r) \geq \frac{r^{\mu+\nu} |a_{\mu\nu}|}{BG(\mu, \nu)}.$$

This gives a lower bound for the growth of $f(z_1, z_2)$ along the hypersurface $g^3 \equiv S_{r=r_0}^1 G^2(r)$, where r varies continuously.

3. **The mapping of the surface $G^2(r)$.** Let us introduce the function

$$(3.1) \quad F(f, \alpha) = e^{e^{-i\alpha} f(z_1, z_2)} = \sum_{r,s=0}^{\infty} A_{rs} z_1^r z_2^s,$$

where $0 \leq \alpha \leq 2\pi$ and f is defined as in the previous sections. The coefficients $\{A_{rs}\}$ are functions of α and a combination of the a_{mn} 's such that $m \leq r$ and $n \leq s$. We define the region $R^2(r)$ as the product of the half-planes

$$(3.2) \quad f_1 \cos \alpha + f_2 \sin \alpha \leq Q(\alpha, r), \quad 0 \leq \alpha < 2\pi,$$

where f_1 and f_2 are cartesian coordinates in the $f_1 f_2$ -plane, and

$$(3.3) \quad Q(\alpha, r) \equiv \log |A_{\mu\nu}(a, \alpha)| + (\mu + \nu) \log r - \log G(\mu, \nu) - \log B.$$

THEOREM II. *Let $f(z_1, z_2) = f_1 + if_2$. Then the smallest convex domain enclosing the mapping of $G^2(r)$ on the $f_1 f_2$ -plane contains the closed convex region $R^2(r)$ which depends only on the coefficients of the expansion of $f(z_1, z_2)$ and the surface $G^2(r)$.*

This gives a lower bound, so to speak, of the mapping of $G^2(r)$ on the $f_1 f_2$ -plane.

Proof of Theorem II. Let

$$(3.4) \quad P(r) = \max |e^{-i\alpha f(z_1, z_2)}|$$

on the surface $G^2(r)$; then from (3.4) and (2.18)

$$\begin{aligned} \log P(r) &= \log |\exp \{e^{-i\alpha f^*(z_1, z_2)}\}| \\ (3.5) \quad &= \log |\exp \{(f_1^* \cos \alpha + f_2^* \sin \alpha)\}| \\ &\quad \cdot |\exp \{-i(f_1^* \sin \alpha - f_2^* \cos \alpha)\}| \\ &= f_1^* \cos \alpha + f_2^* \sin \alpha \\ (3.6) \quad &\geq \log |A_{\mu\nu}(a, \alpha)| + (\mu + \nu) \log r - \log G(\mu, \nu) - \log B = Q(\alpha, r), \end{aligned}$$

where the * indicates that value of f which gives $|P|$ its maximum, for a given α . Now, for each α , $Q(\alpha, r)$ has a fixed value (depending on r). It is clear from (3.6) that at least one point of the mapping, namely, (f_1^*, f_2^*) , will lie in the half-plane

$$(3.7) \quad f_1 \cos \alpha + f_2 \sin \alpha \geq Q(\alpha, r).$$

The region $R^2(r)$ will therefore be contained in the smallest convex domain containing the mapping of $G^2(r)$ on the $f_1 f_2$ -plane. Theorem II is then proved.

It is clear that a similar theorem will hold for any surface for which we have a lower bound for the maximum of the function $f(z_1, z_2)$ on the surface. For example, we can state similar theorems for the surfaces considered in §§4 and 5.

4. Further properties of the function on other surfaces of the type $G^2(r)$.

Let us consider the finite four-dimensional region $\mathfrak{M}^4(r)$ bounded by the three infinite hypersurfaces:

$$\begin{aligned} s_1^3(r) &\equiv E[z_2 = re^{i\lambda_1}, 0 \leq \lambda_1 \leq 2\pi], \\ (4.1) \quad s_2^3(r) &\equiv E[z_1 = re^{i\lambda_2} + C_2 z_2, 0 \leq \lambda_2 \leq 2\pi], \\ s_3^3(r) &\equiv E[z_1 = re^{i\lambda_3} - C_3 z_2, 0 \leq \lambda_3 \leq 2\pi], \end{aligned}$$

where, as above, r is a parameter and C_2 and C_3 are positive constants less than unity. This restriction on C_2 and C_3 is necessary in order that the hyper-surfaces of (3.1) form the boundary of a finite closed domain. Let $G_{ks}^2(r)$ be that part of $s_k^3(r) \cdot s_s^3(r)$ which belongs to the boundary of $\mathfrak{M}^4$. Now let

$$(4.2) \quad G^2(r) \equiv G_{12}^2(r) + G_{13}^2(r) + G_{23}^2(r).$$

Let also $g^3 \equiv S_{r=r_1}^n G^2(r)$, and $g_{ks}^3 \equiv S_{r=r_1}^n G_{ks}^2(r)$, where r varies continuously and $r_1 < \infty$.

Let $f(z_1, z_2)$, as before, be an analytic function regular in $\overline{\mathfrak{M}}^4$. We now apply Bergman's integral formula⁽³⁾ for functions of two complex variables which states that at a point (t_1, t_2) in $\mathfrak{M}^4$,

$$\begin{aligned} f(t_1, t_2) &= \frac{1}{2} \sum'_{k,s} M_{ks}(t_1, t_2) \\ &= \frac{1}{2(2\pi i)^2} \sum'_{k,s} \iint_{B_{ks}^2} \frac{f(\phi_{ks}^{(1)}, \phi_{ks}^{(2)}) B_{ks}(t_1, t_2, \lambda_k, \lambda_s) d\lambda_k d\lambda_s}{\Phi_k(t_1, t_2, \lambda_k) \Phi_s(t_1, t_2, \lambda_s)}, \end{aligned}$$

$$B_{ks}(t_1, t_2, \lambda_k, \lambda_s) = \frac{Z_{ks}(t_1, t_2, \lambda_k, \lambda_s)}{(\phi_{ks}^{(1)} - t_1)(\phi_{ks}^{(2)} - t_2)}, \quad k \neq s,$$

$$\begin{aligned} Z_{ks}(t_1, t_2, \lambda_k, \lambda_s) &= \frac{D(\phi_{ks}^{(1)}, \phi_{ks}^{(2)})}{D(\lambda_k, \lambda_s)} [\Phi_s(t_1, t_2, \lambda_s) \Phi_k(t_1, \phi^{(1)}, \lambda_k) \\ &\quad - \Phi_k(t_1, t_2, \lambda_k) \Phi_s(t_1, \phi^{(2)}, \lambda_s)], \end{aligned}$$

where B_{ks}^2 is the surface range of integration. We have in our case

$$(4.3) \quad \begin{aligned} \Phi_1 &= z_2 - re^{i\lambda_1}, \\ \Phi_2 &= z_1 - re^{i\lambda_2} - C_2 z_2, \\ \Phi_3 &= z_1 - re^{i\lambda_3} + C_3 z_2; \end{aligned}$$

$$(4.4) \quad \begin{aligned} \begin{cases} z_1 = \phi_{12}^{(1)} \equiv re^{i\lambda_2} + C_2 re^{i\lambda_1}, \\ z_2 = \phi_{12}^{(2)} \equiv re^{i\lambda_1}, \\ z_1 = \phi_{13}^{(1)} \equiv re^{i\lambda_3} - C_3 re^{i\lambda_1}, \\ z_2 = \phi_{13}^{(2)} \equiv re^{i\lambda_1}, \end{cases} \\ \begin{cases} z_1 = \phi_{23}^{(1)} \equiv \frac{1}{C_2 + C_3} [C_2 re^{i\lambda_3} + C_3 re^{i\lambda_2}], \\ z_2 = \phi_{23}^{(2)} \equiv \frac{1}{C_2 + C_3} [re^{i\lambda_3} - re^{i\lambda_2}]; \end{cases} \end{aligned}$$

⁽³⁾ Bergman [2, p. 97] and [3, p. 861].

and consequently,

$$\begin{aligned}
 (4.5) \quad f(t_1, t_2) &= M_{12}(t_1, t_2) + M_{13}(t_1, t_2) + M_{23}(t_1, t_2) \\
 &= \frac{1}{(2\pi i)^2} \iint_{B_{12}^2} \frac{f(\phi_{12}^{(1)}, \phi_{12}^{(2)})(r^2 e^{i(\lambda_1 + \lambda_2)}) d\lambda_1 d\lambda_2}{(re^{i\lambda_1} - t_2)(re^{i\lambda_2} + C_2 t_2 - t_1)} \\
 (4.6) \quad &+ \frac{1}{(2\pi i)^2} \iint_{B_{13}^2} \frac{f(\phi_{13}^{(1)}, \phi_{13}^{(2)})(r^2 e^{i(\lambda_1 + \lambda_3)}) d\lambda_1 d\lambda_3}{(re^{i\lambda_1} - t_2)(re^{i\lambda_3} - C_3 t_2 - t_1)} \\
 &+ \frac{1}{(2\pi i)^2} \iint_{B_{23}^2} \frac{f(\phi_{23}^{(1)}, \phi_{23}^{(2)})(r^2 e^{i(\lambda_2 + \lambda_3)}) d\lambda_2 d\lambda_3}{(re^{i\lambda_2} + C_2 t_2 - t_1)(re^{i\lambda_3} - C_3 t_2 - t_1)}.
 \end{aligned}$$

As in §2 we have that

$$\begin{aligned}
 (4.7) \quad a_{mn} &= \frac{\partial^{m+n} f(0, 0)}{m!n! \partial t_1^m \partial t_2^n} = \frac{\partial^{m+n} [M_{12}(t_1, t_2) + M_{13}(t_1, t_2) + M_{23}(t_1, t_2)]}{m!n! \partial t_1^m \partial t_2^n} \Big|_{t_1, t_2=0}, \\
 &\frac{\partial^{m+n} M_{12}(t_1, t_2)}{\partial t_1^m \partial t_2^n} \\
 (4.8) \quad &= \frac{m!}{(2\pi i)^2} \iint_{B_{12}^2} \frac{\partial^n}{\partial t_2^n} \left[\frac{f(\phi_{12}^{(1)}, \phi_{12}^{(2)})(r^2 e^{i(\lambda_1 + \lambda_2)})}{(re^{i\lambda_1} - t_2)(re^{i\lambda_2} + C_2 t_2 - t_1)^{m+1}} \right] d\lambda_1 d\lambda_2 \\
 &= \frac{m!n!}{(2\pi i)^2} \iint_{B_{12}^2} \frac{f(\phi_{12}^{(1)}, \phi_{12}^{(2)})(r^2 e^{i(\lambda_1 + \lambda_2)}) d\lambda_1 d\lambda_2}{[-\Phi_1(t_2)]^{n+1} [-\Phi_2(t_1, t_2)]^{m+1}},
 \end{aligned}$$

so that

$$\begin{aligned}
 (4.9) \quad \frac{\partial^{m+n} M_{12}(0, 0)}{\partial t_1^m \partial t_2^n} &= \frac{m!n!}{(2\pi i)^2} \iint_{B_{12}^2} \frac{f(\phi_{12}^{(1)}, \phi_{12}^{(2)})(r^2 e^{i(\lambda_1 + \lambda_2)})}{r^{m+n+2} e^{i(m\lambda_2 + n\lambda_1 + \lambda_2 + \lambda_1)}} \\
 &\cdot \sum_{\nu=0}^n \frac{(m+\nu)!}{m!\nu!} (C_2 e^{i(\lambda_1 - \lambda_2)})^\nu d\lambda_1 d\lambda_2.
 \end{aligned}$$

This yields, by a process analogous to that used in §2,

$$(4.10) \quad \frac{1}{m!n!} \left| \frac{\partial^{m+n} M_{12}(0, 0)}{\partial t_1^m \partial t_2^n} \right| \leq \frac{B_{12}(g_{12}^3) M(r) G_{12}(m, n)}{r^{m+n}},$$

where $M(r)$ is the maximum-modulus of f on $G^2(r)$, $B(g_{12}^3)$ is a constant depending on the hypersurface $g_{12}^3 = S_{r=0}^1 G_{12}^2(r)$ and $G_{12}(m, n)$ is a function of m and n , also depending on g_{12}^3 and is defined in a way similar to $G(m, n)$ of §2.

In the same way we obtain

$$(4.11) \quad \frac{1}{m!n!} \left| \frac{\partial^{m+n} M_{13}(0, 0)}{\partial t_1^m \partial t_2^n} \right| \leq \frac{B_{13}(G_{13}^2) M(r) G_{13}(m, n)}{r^{m+n}}.$$

From (4.3), we have

$$(4.12) \quad \Phi_2(t_1, t_2) = (t_1 - re^{i\lambda_2} - C_2 t_2), \quad \Phi_3(t_1, t_2) = (t_1 - re^{i\lambda_3} + C_3 t_2).$$

Hence

$$(4.13) \quad \frac{\partial^m}{\partial t_1^m} \left[\frac{1}{\Phi_2 \Phi_3} \right] = (-1)^m m! \sum_{\nu=0}^m \frac{1}{\Phi_2^{\nu+1} \Phi_3^{m-\nu+1}},$$

and

$$(4.14) \quad \frac{\partial^m}{\partial t_1^m} \left[\frac{1}{\Phi_2 \Phi_3} \right] = (-1)^{m+n} m! n! \sum_{\nu=0}^m \sum_{\mu=0}^n \frac{(m+n-\nu-\mu)! C_2^\mu C_3^{n-\mu}}{(m-\nu)!(n-\mu)! \Phi_2^{\nu+\mu+1} \Phi_3^{m+n-\nu-\mu+1}}$$

$$(4.15) \quad = \frac{(-1)^{m+n} m! n!}{\Phi_3^{m+n+2}} \sum_{\mu=0}^n \frac{1}{(n-\mu)!} \left(\frac{C_2}{C_3} \frac{\Phi_3}{\Phi_2} \right)^\mu$$

$$\cdot \sum_{\nu=0}^m \frac{(m+n-\nu-\mu)!}{(m-\nu)!} \left(\frac{\Phi_3}{\Phi_2} \right)^\nu.$$

Then

$$(4.16) \quad \frac{1}{m!n!} \left| \frac{\partial^{m+n} M_{23}(0, 0)}{\partial t_1^m \partial t_2^n} \right| \leq \frac{1}{4\pi^2} \iint_{B_{23}^2} \frac{|f(\phi_{23}^{(1)}, \phi_{23}^{(2)})|}{r^{m+n}} \sum_{\mu=0}^n \frac{1}{(n-\mu)!} \left(\frac{C_2}{C_3} \right)^\mu$$

$$\cdot \sum_{\nu=0}^m \frac{(m+n-\nu-\mu)!}{(m-\nu)!} d\lambda_2 d\lambda_3$$

$$\leq \frac{B_{23}(G_{23}^2) M(r)}{r^{m+n}} \sum_{\mu=0}^n \frac{1}{(n-\mu)!} \left(\frac{C_2}{C_3} \right)^\mu$$

$$\cdot \sum_{\nu=0}^m \frac{(m+n-\nu-\mu)!}{(m-\nu)!}.$$

The constant $B_{23}(g_{23}^3)$ is given by $(1/4\pi^2) \iint_{B_{23}^2} d\lambda_2 d\lambda_3$, where the precise limits of integration are obtained by a tedious process and can be omitted here since they are not necessary for our purpose; it may be noted, however, that $0 < B_{23} < 1$. We shall denote by $G_{23}(m, n)$ the expression

$$(4.17) \quad \sum_{\mu=0}^n \frac{1}{(n-\mu)!} \left(\frac{C_2}{C_3} \right)^\mu \sum_{\nu=0}^m \frac{(m+n-\nu-\mu)!}{(m-\nu)!}.$$

This gives

$$(4.18) \quad |a_{mn}| \leq \frac{1}{m!n!} \left(\left| \frac{\partial^{m+n} M_{12}(0, 0)}{\partial t_1^m \partial t_2^n} \right| + \left| \frac{\partial^{m+n} M_{13}(0, 0)}{\partial t_1^m \partial t_2^n} \right| + \left| \frac{\partial^{m+n} M_{23}(0, 0)}{\partial t_1^m \partial t_2^n} \right| \right)$$

$$\leq \frac{M(r)}{r^{m+n}} \frac{1}{2} \sum_{k,s=1}^3 B_{ks}(g_{ks}^3) G_{ks}(m, n).$$

Therefore

$$(4.19) \quad M(r) \geq \frac{r^{m+n} |a_{mn}|}{\frac{1}{2} \sum_{k,s=1}^3 B(g_{ks}) G_{ks}(m, n)}.$$

Those values of μ and ν which make the right-hand side of (4.19) a maximum can be obtained by a process similar to that employed in §2. Hence

$$(4.20) \quad M(r) \geq \frac{r^{\nu+\mu} |a_{\mu\nu}|}{\frac{1}{2} \sum_{k,s=1}^3 B_{ks}(g_{ks}^3) G_{ks}(\mu, \nu)}.$$

We can then state

THEOREM III. *Given a function*

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n$$

regular in the closed domain $\overline{\mathfrak{M}^4}(r)$; then along

$$g^3 = \underset{r=r_0}{S}^{r_1} G^2(r) = \underset{r=r_0}{S}^{r_1} (G_{12}^2(r) + G_{13}^2(r) + G_{23}^2(r)),$$

$$M(r) \geq \frac{r^{\nu+\mu} |a_{\mu\nu}|}{\frac{1}{2} \sum_{k,s=1}^3 B_{ks}(g_{ks}^3) G_{ks}(\mu, \nu)}.$$

5. Properties for the function on certain classes of surfaces lying in the boundary and different from $G^2(r)$. We next wish to consider the growth of the function $f(z_1, z_2)$ over a special class of surfaces $H^2(r)$ belonging to the boundary of $\mathfrak{M}^4$. Let

$$(5.1) \quad H^2(r) \equiv E[z_1 = \zeta(r, \lambda_1, \sigma), z_2 = re^{i\lambda_1}, \lambda_1^{(1)} \leq \lambda_1 \leq \lambda_1^{(2)}, \sigma_1 \leq \sigma \leq \sigma_2]$$

where for all σ satisfying $\sigma_1 \leq \sigma < \sigma_2$, and for any fixed λ_1 in the range considered,

$$(5.2) \quad \begin{aligned} \zeta(r, \lambda_1, \sigma) &\in \mathfrak{I}_1^2(r, \lambda_1), \\ \mathfrak{I}_1^2(r, \lambda_1) &\equiv A_1^2(r, \lambda_1) \cdot A_2^2(r, \lambda_1); \\ A_1^2(r, \lambda_1) &\equiv E[|z_1 - C_2 z_2| \leq r, z_2 = re^{i\lambda_1}], \\ A_2^2(r, \lambda_1) &\equiv E[|z_1 + C_3 z_2| \leq r, z_2 = re^{i\lambda_1}]; \end{aligned}$$

and for $\sigma = \sigma_2$, with λ_1 again fixed,

$$\zeta(r, \lambda_1, \sigma_2) \in s_1^1(r, \lambda_1),$$

where s_1 is the boundary of $\mathfrak{I}_1^2(r, \lambda_1)$. It will be assumed that the set of all points of $H^2(r)$ for which λ_1 has an arbitrary fixed value in the range consid-

ered is a continuous curve $h^1(r)$ ⁽⁴⁾ with an initial point $z_1 = \zeta(r, \lambda_1, \bar{\sigma}_1(\lambda_1))$, and a terminal point on $s_1^1(r, \lambda_1)$. The surface $H^2(r)$ lies completely in that part of $s_1^3(r)$ which belongs to the boundary of the $\mathfrak{M}^4(r)$ of the previous section. A portion of the boundary of $H^2(r)$ lies on $G^2(r)$ of (4.2).

Let the maximum-modulus of $f(z_1, z_2)$ on $H^2(r)$ be $\gamma(r)$. We now map (using for simplicity the same notation for the mapped region) $\mathfrak{J}_1^2$ into the unit circle so that $z_1=0$ goes into itself and the direction of the real axis, $\Re(z_1)=0$, at the point $z_1=0$, remains unchanged. The curve $h^1(r)$ maps into a segment of a continuous curve, its initial point determined by $\sigma = \bar{\sigma}_1(\lambda_1)$ and its terminal point lies on the unit circle. Now let $\theta = |z_1| = |\zeta(r, \lambda_1, \bar{\sigma}_1(\lambda_1))|$ for $\lambda_1^{(1)} \leq \lambda_1 \leq \lambda_1^{(2)}$. The quantities θ and $\alpha = \lambda_1^{(2)} - \lambda_1^{(1)}$ were introduced by Bergman and are the characteristic numbers of the surface⁽⁵⁾.

One form of the Milloux theorem is⁽⁶⁾: Let J be a continuous finite arc lying in the unit circle $|z| \leq 1$ joining a point z_0 within the circle to a point on the boundary. Let $W(z)$ be regular, single-valued, and $|W(z)| < 1$ inside the unit circle, and let $|W(z)| \leq \omega$ on J . Then

$$(5.3) \quad |W(0)| < \omega^{(2/\pi) \sin^{-1}(1-\theta)/(1+\theta)},$$

where $\theta' = |z_0|$.

Using this theorem for the mapped region $\mathfrak{J}_1^2$ with

$$(5.4) \quad W(z_1) = \frac{f(z_1, z_2^*)}{M(r)},$$

we have

$$(5.5) \quad |W(z_1)| = \frac{|f(z_1, z_2^*)|}{M(r)} \leq \frac{\gamma(r)}{M(r)} < 1,$$

and get, letting $\Theta = (2/\pi) \sin^{-1}(1-\theta)/(1+\theta)$,

$$(5.6) \quad |f(0, z_2)| < M^{1-\Theta} \gamma^\Theta,$$

where $M(r)$ is the maximum-modulus of $f(z_1, z_2)$ in $\overline{\mathfrak{M}}^4$, λ_1^* is an arbitrarily chosen value of λ_1 in the range considered, and $z_2 = z_2^* = r e^{i\lambda_1^*}$.

Now

$$(5.7) \quad a_{0n} = \frac{1}{2\pi i} \left\{ \int_0^{\lambda_1^{(1)}} \frac{f(0, r e^{i\lambda_1})}{r^n e^{in\lambda_1}} d\lambda_1 + \int_{\lambda_1^{(1)}}^{\lambda_1^{(2)}} \frac{f(0, r e^{i\lambda_1})}{r^n e^{in\lambda_1}} d\lambda_1 + \int_{\lambda_1^{(2)}}^{2\pi} \frac{f(0, r e^{i\lambda_1})}{r^n e^{in\lambda_1}} d\lambda_1 \right\},$$

⁽⁴⁾ The restriction that $h^1(r)$ be continuous is not essential since theorems of the Milloux type hold for more general one-dimensional sets.

⁽⁵⁾ Bergman [1, pp. 347-348, Corollary]; and [4, pp. 200-201].

⁽⁶⁾ R. Nevanlinna [5].

$$(5.8) \quad |a_{0n}| \leq \frac{1}{2\pi} \left\{ \int_0^{\lambda_1^{(1)}} \frac{|f(0, re^{i\lambda_1})|}{r^n} d\lambda_1 + \int_{\lambda_1^{(1)}}^{\lambda_1^{(2)}} \frac{|f(0, re^{i\lambda_1})|}{r^n} d\lambda_1 \right. \\ \left. + \int_{\lambda_1^{(2)}}^{2\pi} \frac{|f(0, re^{i\lambda_1})|}{r^n} d\lambda_1 \right\},$$

and

$$(5.9) \quad |a_{0n}| \leq \frac{1}{2\pi r^n} \left[(2\pi - \alpha) \overline{M} + \alpha M \left(\frac{\gamma}{M} \right)^\Theta \right],$$

where $\overline{M} = \max |f(0, z_2)| = \max |\sum_{\nu=0}^{\infty} a_{0\nu} z_2^\nu|$. Then

$$(5.10) \quad \frac{2\pi}{\alpha} \left[\frac{|a_{0n}| r^n}{M} - \frac{\overline{M}}{M} \right] + \frac{\overline{M}}{M} \leq \left(\frac{\gamma}{M} \right)^\Theta.$$

Let Δ be defined by the equation

$$(5.11) \quad \Delta = 1 - \frac{|a_{0\mu}| r^\mu}{\overline{M}},$$

μ being that n which maximizes $|a_{0n}| r^n$, and μ depends on r . Then Δ is positive.

If $\alpha > 2\pi\Delta$, we have that

$$(5.12) \quad \gamma(r) \geq M(r) \left[\frac{\overline{M}}{M} \left(1 - \frac{2\pi}{\alpha} \Delta \right) \right]^{\Theta^{-1}(\theta)},$$

where the right-hand side is positive.

From these results we can state

THEOREM IV. *Given the function*

$$f(z_1, z_2) = \sum_{m,n=0}^{\infty} a_{mn} z_1^m z_2^n,$$

regular in $\overline{\mathfrak{M}}^4(r)$. Let $\max |f(z_1, z_2)| \leq \gamma(r)$ on the surface $H^2(r)$ of (5.1) having the characteristic numbers $\theta(r)$ and $\alpha, \alpha = \lambda_1^{(2)} - \lambda_1^{(1)} > 2\pi\Delta$, where $\Delta \equiv 1 - |a_{0\mu}| r^\mu / \overline{M}$; then

$$(5.13) \quad \gamma(r) \geq M(r) \left[\frac{\overline{M}(r)}{M(r)} \left(1 - \frac{2\pi\Delta}{\alpha} \right) \right]^{\Theta^{-1}(\theta)},$$

where $M = \max |f(z_1, z_2)|$ and $\overline{M} = \max |f(0, z_2)|$.

Since

$$(5.14) \quad \frac{|a_{0\mu}| \rho^{\mu+1}}{\rho - r} > \overline{M}(r), \quad \rho > r,$$

a lower bound for $\gamma(r)$ can be obtained in terms of the coefficients of $f(z_1, z_2)$ by replacing in (5.12) $M(r)$ by the right-hand side of (4.20), $\overline{M}(r)$ by $|a_{0\mu}|r^\mu$, and the $\overline{M}(r)$ in Δ by

$$(5.15) \quad \frac{|a_{0\mu'}| \rho_1^{\mu'+1}}{\rho_1 - r}$$

where

$$(5.16) \quad |a_{0\mu'}| \rho_1^{\mu'} = \text{l.u.b.}_{\substack{1+\epsilon < \rho < \infty}} \left[\max_n |a_{0n}| \rho^n \right],$$

for an arbitrary positive ϵ . ρ_1 is a function of r and of the coefficients $\{a_{0n}\}$, and can be determined by a process similar to the Newton polygon method.

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